

YANGIAN SYMMETRY IN D=4 SUPERCONFORMAL YANG-MILLS THEORY*

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We will discuss an integrable structure for weakly coupled superconformal Yang-Mills theories, describe certain equivalences for the Yangian algebra, and fill a technical gap in our previous study of this subject.

1. Symmetry Generators and Anomalous Dimensions

In [1] it was shown that the classical Green-Schwarz superstring action for $AdS^5 \times S^5$ possesses a hierarchy of non-local symmetries of the type that exist in integrable field theories [2, 3]. This is due to the fact that the Green-Schwarz superstring in $AdS^5 \times S^5$ can be interpreted as a coset theory where the fields take values in the coset superspace

$$\frac{PSU(2, 2|4)}{SO(4, 1) \times SO(5)} . \quad (1)$$

This coset theory (even though the target is not a symmetric space [4]) admits non-local currents which give rise to charges satisfying a Yangian

*QTS3 Conference Proceedings, University of Cincinnati, September 2003.

[†]Work partially supported by DOE grant DE-FG02-03ER-41262.

[‡]This work is supported by NSF Grants PHY-0140311 and PHY-0243680.

[§]This work is supported by NSF Grant PHY-0070928

algebra. A Yangian algebra $Y(G)$ is an associative Hopf algebra [5, 6, 2, 7] generated by the elements J^A and Q^B with

$$[J^A, J^B] = f_C^{AB} J^C, \quad [J^A, Q^B] = f_C^{AB} Q^C, \quad (2)$$

and the Serre relations

$$\begin{aligned} & [Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] \\ &= \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\}, \end{aligned} \quad (3)$$

$$\begin{aligned} & [[Q^A, Q^B], [J^C, Q^D]] + [[Q^C, Q^D], [J^A, Q^B]] \\ &= \frac{1}{24} (f^{AGL} f^{BEM} f^{KFN} f_{LMN} f_K^{CD} \\ & \quad + f^{CGL} f^{DEM} f^{KFN} f_{LMN} f_K^{AB}) \{J_G, J_E, J_F\}, \end{aligned} \quad (4)$$

for J^A taking values in the Lie algebra of an arbitrary semi-simple Lie group G . (Lie algebra indices A, B, C are raised and lowered with an invariant, nondegenerate metric tensor g_{AB} or g^{AB} .) The symbol $\{A, B, C\}$ denotes the symmetrized product of three operators A, B , and C . Under repeated commutators, the Q^A generate an infinite-dimensional symmetry algebra that has been called the Yangian. The Yangian has a basis \mathcal{J}_n^A where $\mathcal{J}_0^A = J^A$, $\mathcal{J}_1^A = Q^A$, and \mathcal{J}_n^A is an n -local operator that arises in the $(n-1)$ -form commutator of Q 's. Since we will work in this paper mainly with the generators J^A and Q^A , we have given them those special names. The Yangian relations as written above are redundant in the following sense. For $SU(2)$ the relation (3) is trivial. For other cases such as $SU(N)$ with $N \geq 3$, the relation (3) implies the following one (4).

In the superstring on $AdS^5 \times S^5$, the J^A will be generators of $G = PSU(2, 2|4)$, the brackets will generalize to denote either commutators or anticommutators, and f_{AB}^C become the structure constants of $PSU(2, 2|4)$. The Q^A are the new non-local charges whose existence was found in [1]. If the AdS/CFT correspondence is correct, then on the CFT side we must have the same Yangian symmetry. The question arises of what could be the Q^A charges in the super Yang-Mills side. We will answer this question in the extreme weak coupling limit, that is, the opposite limit from that which is considered in [1]. In order to justify our guess for Q^A , we will review how non-local symmetries arise in two-dimensional sigma models.

Let us consider a model with a group G of symmetries; the Lie algebra of G has generators T_A obeying $[T_A, T_B] = f_{AB}^C T_C$. The action of G is generated by a current $j^\mu{}^A$ that is conserved, $\partial_\mu j^\mu{}^A = 0$. Non-local charges arise if, in addition, the Lie algebra valued current $j_\mu = \sum_A j_\mu^A T_A$ can be

interpreted as a flat connection,

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0. \quad (5)$$

(Indices of j_μ are raised and lowered using the Lorentz metric in two dimensions.) The conservation of j_μ leads in the usual fashion to the existence of conserved charges that generate the action of G :

$$J^A = \int_{-\infty}^{\infty} dx j^{0A}(x, t). \quad (6)$$

In addition, a short computation using (5) reveals that

$$Q^A = f_{BC}^A \int_{-\infty}^{\infty} dx \int_x^{\infty} dy j^{0B}(x, t) j^{0C}(y, t) - 2 \int_{-\infty}^{\infty} dx j_1^A(x, t) \quad (7)$$

is also conserved. The charges J^A and Q^A generate a Yangian algebra [2, 3], even though the infinitesimal transformations generated by them generate half of a Kac-Moody algebra [8].

There are also discrete spin systems, that is systems in which the dynamical variables live on a one-dimensional lattice rather than on the real line, that similarly have Yangian symmetry. The lattice definition of J^A is clear. We assume that the spins at each site i have G symmetry, and transform in some representation \mathcal{R} . We let J_i^A be the symmetry operators at the i^{th} site. The total charge generator for the whole system is then

$$J^A = \sum_i J_i^A. \quad (8)$$

What about Q^A ? For general G and \mathcal{R} , there is no satisfactory definition of Q^A . However, for $G = SU(N)$, a Q^A can be defined for any \mathcal{R} . For certain representations, the requisite formula for Q^A is particularly simple. One just takes the obvious discretization of the bilocal part of (7):

$$Q^A = f_{BC}^A \sum_{i < j} J_i^B J_j^C. \quad (9)$$

This is the right formula in many of the most commonly studied lattice integrable systems.

In (9), one has made no attempt to discretize the second term in (7). In fact, for many choices of \mathcal{R} , discretizing that second term is impossible for elementary reasons. One would expect a discretization of $\int dx j_1^A$ to be of the form $\sum_i j_i^A$ where j_i^A acts on the i^{th} site and transforms in the adjoint representation. If \mathcal{R} is such that the adjoint representation of G appears only once in the decomposition of $\mathcal{R} \otimes \overline{\mathcal{R}}$, then j_i^A would have

to be a multiple of J_i^A . This is so, for example, if \mathcal{R} is the fundamental representation of $SU(N)$ or $SU(N|M)$ (and more generally if \mathcal{R} is the representation of k^{th} rank antisymmetric tensors, for any k). But taking j_i^A to be a multiple of J_i^A just adds to Q^A a multiple of J^A ; this is an outer automorphism of the Yangian algebra, and so does not help (or hurt) in obeying the Serre relations. We will argue that (9) is the correct formula for the Q^A in the weak coupling limit of Yang-Mills theory. This will fill a technical gap in our previous paper [9], where we showed that Q^A as defined by this formula commutes with the one-loop anomalous dimension operator, leading to an infinity of conservation laws, but we did not show that it obeys the Serre relations, which is needed to ensure that the resulting integrable structure is the standard Yangian algebra.

Another justification for the proposal (9) is that it has an analog in gauge field theory at $g^2N = 0$. In order to make contact with conventional Noether current symmetry analysis, we give the expression for the non-local charge (9) in terms of the elementary fields of the super Yang-Mills Lagrangian

$$\mathcal{L} = \frac{1}{g_{YM}^2} \text{Tr} \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^I D^\mu \phi^I - \frac{1}{2} [\phi^I, \phi^J] [\phi^I, \phi^J] + \text{fermions} \right). \quad (10)$$

For simplicity, we will only consider $A \in so(2, 4)$. In the classical theory, the symmetry currents for the conformal group are given in terms of the improved energy-momentum tensor by

$$j^{A\mu}(x) = \kappa_\nu^A \theta^{\mu\nu}(x), \quad (11)$$

where κ_μ^A are the conformal Killing vectors, and

$$\begin{aligned} \theta^{\mu\nu} = & 2\text{Tr} F^{\mu\rho} F_\rho^\nu + 2\text{Tr} D^\mu \phi^I D^\nu \phi^I - g^{\mu\nu} \mathcal{L} - \frac{1}{3} \text{Tr} (D^\mu D^\nu - g^{\mu\nu} D_\rho D^\rho) \phi^I \phi^I \\ & + \text{fermions}. \end{aligned} \quad (12)$$

The currents (11) are conserved at any g^2N using the classical interacting equations of motion. If we set $g^2N = 0$, we note that the *untraced* matrix

$$\begin{aligned} (\theta^{\mu\nu})_i^j = & F^{\mu\rho} F_\rho^\nu + F^{\nu\rho} F_\rho^\mu + \partial^\mu \phi^I \partial^\nu \phi^I \\ & + \partial^\nu \phi^I \partial^\mu \phi^I - g^{\mu\nu} \left(\frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} + \partial_\mu \phi^I \partial^\mu \phi^I \right) \\ & - \frac{1}{3} (\partial^\mu \partial^\nu - g^{\mu\nu} \partial_\rho \partial^\rho) \phi^I \phi^I + \text{fermions} \end{aligned} \quad (13)$$

is also conserved, as is $\kappa_\nu^A (\theta^{\mu\nu})_i^j$. Here i, j are the matrix labels of the gauge group generators $(T^A)_i^j$. It follows that we can construct non-local

conserved charges by

$$Q_0^{AB\dots} = \int_M \kappa_\nu^A(\theta^{0\nu})_i^j \int_M \kappa_\rho^B(\theta^{0\rho})_j^k \dots, \quad (14)$$

where M is an initial value surface in spacetime. In free field theory, this acts on a chain of partons rather as (9) does, but we have no idea how to extend the definition to $g^2N \neq 0$.

Inspired by the above examples, our basic assumption in [9] is that in $\mathcal{N} = 4$ super Yang-Mills theory at $g^2N = 0$, with J_i^A understood as the $PSU(2, 2|4)$ generators of the i^{th} parton, (9) is the correct formula for the Yangian generators Q^A . Our assumption, in other words, is that the bilocal symmetry deduced from [1] goes over to (9) for $g^2N \rightarrow 0$. Of course, in any case (8) is the appropriate free field formula for the J^A , so we do not need to state any hypothesis for these generators. And no further assumption is needed for the higher charges in the Yangian; they are generated by repeated commutators of the Q^A . So our hypothesis about Q^A completely determines the form of the Yangian generators in the free-field limit.

Though our goal in these notes is to fill the above-mentioned technical gap in the previous analysis, for completeness we here sketch some of the reasoning in our previous paper. (We return in the next section to the question of why the simple bilocal formula does give a representation of the Yangian.) Having made an ansatz for how the Yangian algebra is realized at $g^2N = 0$, we consider what happens when g^2N is not quite zero. Some generators of the Yangian do not receive quantum corrections. For example, the spatial translation symmetries and the Lorentz generators are uncorrected, because the theory can be regularized in a way that preserves them. But the dilatation operator D – the generator of scale transformations – certainly is corrected. The corrections to the eigenvalues of D are called anomalous dimensions.

We assume, in view of [1], that the $\mathcal{N} = 4$ Yang-Mills theory in the planar limit does have Yangian symmetry for all g^2N . If so, the corrections modify the form of the generators, but preserve the commutation relations. One of the commutation relations says that Q^A transforms in the adjoint representation of the global group $PSU(2, 2|4)$ generated by J^A : $[J^A, Q^B] = f^{ABC}Q^C$. We will write J^A and Q^A for the charges at $g^2N = 0$, and δJ^A and δQ^A for the corrections to them of order g^2N . We write \tilde{J}^A and \tilde{Q}^A for the exact generators (which depend on g^2N), so $\tilde{J}^A = J^A + (g^2N)\delta J^A + \mathcal{O}((g^2N)^2)$, and likewise for \tilde{Q}^A . To preserve the commutation relations, we have

$$[\delta J^A, Q^B] + [J^A, \delta Q^B] = f^{ABC} \delta Q^C. \quad (15)$$

We are now going to make an argument for the Yangian that parallels one used in [10] for the $PSU(2, 2|4)$ generators. We consider the special case of this relation in which A is chosen so that J^A is the dilatation operator D . We also pick a basis Q^B of the Q 's to diagonalize the action of D , so the $PSU(2, 2|4)$ algebra reads in part $[D, Q^B] = \lambda^B Q^B$, where λ^B is the bare conformal dimension of Q^B . Then (15) gives us

$$[\delta D, Q^B] + [D, \delta Q^B] = \lambda^B \delta Q^B. \quad (16)$$

However, in perturbation theory, operators only mix with other operators of the same classical dimension. So just as $[D, Q^B] = \lambda^B Q^B$, we have $[D, \delta Q^B] = \lambda^B \delta Q^B$. Combining this with (15), we have therefore

$$[\delta D, Q^B] = 0. \quad (17)$$

Precisely the same argument was used in [10] to show that $[\delta D, J^A] = 0$; this was a step in determining δD . Combining this with (17), we see that δD must commute with the $g^2 N = 0$ limit of the whole Yangian.

The structure of perturbation theory implies in addition that the operator δD is a sum of operators local along the chain; this fact has been exploited in [11] and many subsequent papers. (In fact, δD , as described explicitly in [10], is a sum of operators that act on nearest neighbor pairs.) The operators of this type that commute with the Yangian – where here we mean the Yangian representation most commonly studied in lattice integrable models, which for us is the one generated at $g^2 N = 0$ by J^A and Q^A – are called the Hamiltonians of the integrable spin chain. Thus, from our assumption about the free-field limit of the Yangian, we are able, starting with the nonlocal symmetries found in [1], to deduce the basic conclusion of [12], found earlier in a special case in [13], that δD is a Hamiltonian of an integrable spin chain.

In our paper [9], we verify this picture by proving directly, using formulas for the one loop operator computed in [10], [12], that it is true that δD commutes with the Yangian. Since its commutativity with J^A was already used in [10] to compute δD , the only novelty is to verify that $[\delta D, Q^A] = 0$.

From what we have said, it is clear that the appearance of a Hamiltonian that commutes with the Yangian depends on expanding to first order near $g^2 N = 0$. In the exact theory, at a nonzero value of $g^2 N$, one would simply say that the exact dilatation operator \mathcal{D} , which of course depends on $g^2 N$,

is one of the generators of the Yangian. It is not the case in the exact theory that one has a Yangian algebra and also a dilatation operator that commutes with it.

This result is highly non trivial, and is heavily based on non-trivial properties of the loop correction to the dilaton operator. Hence, it is a good step in the direction of proving that (9) is the correct Yangian charge. To strengthen our guess for the Q^A charges as expressed in (9), we will show that they satisfy the Serre relation, or equivalently the nesting relation given in (3). We first explain the key steps for $SU(N)$, and then outline how these steps change and generalize in the case of $PSU(2, 2|4)$ under consideration.

2. Yangian Relations for $SU(N)$

We return to the question of showing that the bilinear ansatz (9) does give, under certain conditions, a solution of the Serre relations. We will show explicitly that (as indicated in [14]) the standard relations for the Yangian $Y(G)$, which are valid for any Lie group G , are equivalent when G is $SU(N)$ (or $U(N)$) to a matrix form of the commutation relations. We then use this to show that for certain types of representation \mathcal{R} , the formula (9) does give a solution of the Serre relations.

For $G = SU(N)$, the Lie algebra is the space of traceless $N \times N$ matrices. Instead of describing the Lie algebra in terms of an abstract basis J^A , as one could do for any Lie group G , it is useful for $SU(N)$ to describe it in terms of generators J^a_b , $a, b = 1, \dots, N$, with $\sum_a J^a_a = 0$, and obeying

$$\begin{aligned} [J^a_b, J^c_d] &= \delta^c_b J^a_d - \delta^a_d J^c_b, \\ [J^a_b, Q^c_d] &= \delta^c_b Q^a_d - \delta^a_d Q^c_b. \end{aligned} \quad (18)$$

Similarly, the generators Q^A of the Yangian are rewritten as a traceless $N \times N$ matrix of operators Q^a_b . The Serre relation then becomes

$$\begin{aligned} &[J^a_b, [Q^c_d, Q^e_f]] - [Q^a_b, [J^c_d, Q^e_f]] \\ &= \frac{h^2}{4} \sum_{p,q} ([J^a_b, [J^c_p J^p_d, J^e_q J^q_f]] - [J^a_p J^p_b, [J^c_d, J^e_q J^q_f]]). \end{aligned} \quad (19)$$

This matrix form of the Yangian relations is the most familiar one in integrable systems, and will be useful in the generalization to the superalgebra. We will prove below that (19) is equivalent to (3).

To do this, it is useful to work out in more detail how general Lie algebra notation simplifies for $SU(N)$. The generators T^A , $A = 1, \dots, N^2 - 1$,

of $SU(N)$ can be regarded as $N \times N$ matrices in the fundamental representation of $SU(N)$. We use the conventions $[T^A, T^B] = f_C^{AB} T^C$, $f_{ABC} f_{ABE} = 2N \delta_{CE}$,

$$\begin{aligned} \text{Tr} T^A T^B &= -\delta^{AB} \\ \text{Tr} T^A T^B T^C &= -\frac{1}{2}(f_{ABC} - id_{ABC}). \end{aligned} \quad (20)$$

We define the totally symmetric invariant tensor $id_{ABC} = \text{Tr}(\{T^A, T^B\} T^C)$, and use the fact that the generators in this representation, together with the identity matrix, span the space of all complex $N \times N$ matrices. We get:

$$\begin{aligned} T^A T^B &= \frac{1}{2}(f_{ABC} T^C + \{T^A, T^B\}) = -\frac{1}{N} \delta_{AB} + \frac{1}{2}(f_{ABC} - id_{ABC}) T^C \\ [T^A, T^B] &= f_{ABC} T^C \\ \{T^A, T^B\} &= -\frac{2}{N} \delta_{AB} - id_{ABC} T^C. \end{aligned} \quad (21)$$

Explicitly, we write the matrix elements of the matrix T^A , in the fundamental representation, as T^{Aa}_b , $a, b = 1 \dots N$. We define

$$J^A = -T^{Ab}_a J^a_b, \quad Q^A = -T^{Ab}_a Q^a_b, \quad (22)$$

where the two-index generators in (22) are traceless $\sum_a J^a_a = 0 = \sum_a Q^a_a$, and we can invert $J^a_b = J^A T^{Aa}_b$ and $Q^a_b = Q^A T^{Aa}_b$. Note that although T^{Aa}_b is in the N of $SU(N)$, J^A and consequently J^a_b can be in an arbitrary representation. The relation (19), which is sometimes called the nesting relation, reduces to (3) as follows. Multiplying (19) by $T^{Ab}_a T^{Bd}_c T^{Cf}_e$, we find

$$\begin{aligned} &[Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] \\ &= -\frac{h^2}{4} ((\text{Tr} T^B T^D T^E) \text{Tr}(T^C T^F T^G) [J^A, [J^D J^E, J^F J^G]] \\ &\quad - (\text{Tr} T^A T^D T^E) \text{Tr}(T^C T^F T^G) [J^D J^E, [J^B, J^F J^G]]) \\ &= \frac{h^2}{16} (-f_{ACG} f_{GLM} (d_{MDE} d_{BLF} + d_{MFED} d_{BLD}) \\ &\quad + f_{ABG} f_{GLM} (d_{MDE} d_{CLF} + d_{MFED} d_{CLD}) \\ &\quad + f_{BCG} f_{GLM} (d_{MDE} d_{ALF} + d_{MFED} d_{ALD})) J^D J^E J^F. \end{aligned} \quad (23)$$

To evaluate the products of d symbols and structure constants solely in terms of the structure constants as in (3), we use the following identities. In

addition to the Jacobi identity, there is a similar formula $[\{T^A, T^B\}, T^C] + [\{T^C, T^A\}, T^B] + [\{T^B, T^C\}, T^A] = 0$, which reduces to

$$d_{ABE}f_{ECD} + d_{CAE}f_{EBD} + d_{BCE}f_{EAD} = 0. \quad (24)$$

Another identity $[[T^A, T^B], T^C] + \{\{T^C, T^A\}, T^B\} - \{\{T^B, T^C\}, T^A\} = 0$ results in

$$f_{ABE}f_{CDE} = \frac{4}{N}(\delta_{AC}\delta_{BD} - \delta_{BC}\delta_{AD}) + d_{ACE}d_{BDE} - d_{BCE}d_{ADE}, \quad (25)$$

and

$$d_{ABC}d_{ABE} = 2(N - \frac{4}{N})\delta_{CE}. \quad (26)$$

From the Jacobi identity, we find the triple product

$$f_{DMA}f_{ABE}f_{ECD} = -Nf_{MBC}, \quad (27)$$

from (24) we find

$$d_{ABE}f_{ECD}f_{DFA} = -Nd_{BCF} \quad (28)$$

and from (25), and $\sum_A d_{AAB} = 0$, we have

$$\begin{aligned} d_{ACE}d_{EBD}f_{DMA} &= (N - \frac{4}{N})f_{CBM} \\ d_{DMA}d_{ACE}d_{EBD} &= (N - \frac{12}{N})d_{MCB}. \end{aligned} \quad (29)$$

Note that (25) expresses the difference of two products of d symbols in terms of the structure constants. We used (24) in deriving (23). It is convenient to symmetrize on the D, E, F indices in (23) and use (28,29). Then (23) becomes

$$\begin{aligned} &[Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] \\ &= \frac{h^2}{48} (f_{ABG}f_{GLM}d_{MDE}d_{CLF} - (A \leftrightarrow C) - (B \leftrightarrow C))\{J^D, J^E, J^F\}, \quad * \\ &= \frac{h^2}{48} ((f_{ALG}f_{GBM} + f_{BLG}f_{GAM})d_{MDE}d_{CLF} - (A \leftrightarrow C) - (B \leftrightarrow C)) \\ &\quad \{J^D, J^E, J^F\} \\ &= \frac{h^2}{24} (((f_{FAL}f_{DBM}d_{CLG}d_{GME} - (A \leftrightarrow B)) - (A \leftrightarrow C) - (B \leftrightarrow C)) \\ &\quad + 2(f_{CAL}f_{DBM}d_{LFG}d_{GME} - (A \leftrightarrow B) - (B \leftrightarrow C)))\{J^D, J^E, J^F\} \\ &= \frac{h^2}{24} (((f_{FAL}f_{DBM}d_{CLG}d_{GME} - (A \leftrightarrow B)) - (A \leftrightarrow C) - (B \leftrightarrow C)) \\ &\quad - (f_{BAL}f_{GLM}d_{CFG}d_{MDE} - (A \leftrightarrow C) - (B \leftrightarrow C)))\{J^D, J^E, J^F\} * \\ &\quad (30) \end{aligned}$$

where $\{J^D, J^E, J^F\}$ is the totally symmetrized product,

$$\begin{aligned} \{J^D, J^E, J^F\} = & J^D J^E J^F + J^E J^D J^F + J^F J^E J^D \\ & + J^E J^F J^D + J^D J^F J^E + J^F J^D J^E. \end{aligned} \quad (31)$$

We observe that the two starred lines are proportional, so we have

$$\begin{aligned} & 3 \frac{h^2}{48} (f_{ABG} f_{GLM} d_{MDE} d_{CLF} - (A \leftrightarrow C) - (B \leftrightarrow C)) \{J^D, J^E, J^F\} \\ &= \frac{h^2}{24} ((f_{FAL} f_{DBM} d_{CLG} d_{GME} - (A \leftrightarrow B)) - (A \leftrightarrow C) - (B \leftrightarrow C)) \\ & \quad \{J^D, J^E, J^F\} \\ &= \frac{h^2}{8} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J^D, J^E, J^F\}. \end{aligned} \quad (32)$$

It follows that

$$\begin{aligned} & [Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] \\ &= \frac{h^2}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\}. \end{aligned} \quad (33)$$

This equation is totally antisymmetric in A, B, C , and for $h = 1$ is the equation (3).

A Useful Criterion

The point of this lengthy analysis is that although it is difficult to find a solution of the Yangian relation (3), it is much easier to find a solution of (19). The basic case is the case of just a single spin. We want to find a criterion under which, for $G = SU(N)$ and some irreducible representation \mathcal{R} at a single site, we can obey the Yangian algebra with the simple choice $Q^A = Q^a_b = 0$. In (3), it is unclear when this works, but in (19), we can easily find a simple criterion.

Since the left hand side of (19) obviously vanishes for $Q^A = 0$, we need a criterion for vanishing of the right hand side. The object $J^c_p J^p_d$ that appears in (19) is a linear combination of pieces that transform as the singlet of $SU(N)$ and the adjoint. If \mathcal{R} is irreducible, the singlet piece is a multiple of the identity and does not contribute in the commutator. If moreover \mathcal{R} is such that the adjoint only appears once in the decomposition of $\mathcal{R} \otimes \overline{\mathcal{R}}$, then (modulo the irrelevant c -number) $J^c_p J^p_d$ is a multiple of J^c_d . Similarly, $J^e_q J^q_f$ can be replaced by the same multiple of J^e_f , and $J^a_p J^p_b$ by the same multiple of J^a_b . Once this is done, the right hand side of (19) vanishes.

So we have a criterion for finding irreducible representations \mathcal{R} of $SU(N)$ such that for a single-spin system, the Yangian algebra is obeyed with $Q^A = Q^a_b = 0$. This criterion is not obeyed for all representations. For example, if \mathcal{R} is the adjoint representation, then the criterion is not obeyed, since the adjoint then appears twice in the decomposition of $\mathcal{R} \otimes \overline{\mathcal{R}}$ (which in this example is the same as $\mathcal{R} \otimes \mathcal{R}$).

However, many important examples arise from representations \mathcal{R} that *do* obey the criterion. Basic examples are the fundamental representation, and more generally the representation of antisymmetric k^{th} rank tensors, for any k .

A Chain Of Spins

Now let us explain how to go from this single-spin result to a representation of the Yangian algebra on a whole chain of spins.

The important property of the Yangian algebra is that it is a Hopf algebra, which means that there is a natural recipe for defining a tensor product of representations. If \mathcal{A} is an algebra, a “coproduct” is a map $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ that is a homomorphism of algebras (and obeys some additional axioms of which we explain the relevant one later). For our purposes, this means that the coproduct maps operators that represent \mathcal{A} in a single-spin representation \mathcal{R} to operators that represent \mathcal{A} in the Hilbert space $\mathcal{R} \otimes \mathcal{R}$ of a two-spin system. Moreover, one can repeat the process, using the homomorphism $\Delta \otimes 1 : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, or alternatively the homomorphism $1 \otimes \Delta : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, to map a representation in the two-spin system to a representation in the three-spin system. (Here, for example, $1 \otimes \Delta$ is the operator that acts as the identity on the first spin and acts by Δ to map the Hilbert space of the second spin in a two-spin system to that of the last two spins in a three-spin system. Similarly $\Delta \otimes 1$ acts as the identity on the second or last spin while mapping the Hilbert space of the first to that of a two-spin system.) Repeating the process, one gets a representation of \mathcal{A} in an n -spin system, for any n . Apart from being a homomorphism of algebras, the key axiom obeyed by Δ is “coassociativity,” $\Delta \otimes 1(\Delta) = 1 \otimes \Delta(\Delta)$, which for our purposes says that starting with a given representation of \mathcal{A} in the single-spin system, the representation that one arrives at in the n -spin system does not depend on the precise route by which one applies these formulas.

For the Yangian, the explicit formula for the coproduct is

$$\begin{aligned}\Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \\ \Delta(Q^A) &= Q^A \otimes 1 + 1 \otimes Q^A + f^A_{BC} J^B \otimes J^C.\end{aligned}\tag{34}$$

Using this coproduct, we can determine, given a single-spin representation of the Yangian with $Q^A = 0$, what the representation should be for a multi-spin system. Consider first the two-spin system. The two-spin representation of J^A is $\Delta(J^A) = J^A \otimes 1 + 1 \otimes J^A$. This is a fancy notation for writing the result that we would expect naively, since $J^A \otimes 1$ and $1 \otimes J^A$ are simply in a more typical physics notation the operators J_1^A and J_2^A that act by J^A on the first or second spin. So $\Delta(J^A) = J_1^A + J_2^A$, saying simply that the group generators of the two-spin system are the sum of the single-particle generators. If the single-spin representation of Q^A is zero, then the two-spin representation of Q^A reduces to $\Delta(Q^A) = f_{BC}^A J_1^B \otimes J_2^C$, which in the alternative notation is $f_{BC}^A J_1^B J_2^C$. This agrees with the two-spin case of (9). More generally, by repeated application of the coproduct, one learns that whenever for the one-spin system one can obey the Yangian algebra with $Q^A = 0$, the formula (9) supplies a representation of the Yangian algebra for a chain of spins.

If the adjoint representation appears more than once in the decomposition of $\mathcal{R} \otimes \overline{\mathcal{R}}$, then for the one-spin system, one cannot generally obey the Yangian algebra with $Q^A = 0$. However, for $SU(N)$, the form (19) of the relations implies that one can always, for any representation \mathcal{R} of $SU(N)$, obey the Yangian algebra with $Q^a_b = J^a_p J^p_b/2$. This contrasts with the situation for more general symmetry groups G , where for generic \mathcal{R} there is no choice of Q^A that obeys the Serre relation.

3. Yangian Superalgebra

The analysis in the previous section would have worked in just the same way if we replace the simple group $SU(N)$ by the non-simple group $U(N)$. It similarly works for the supergroup $U(N|M)$, and for $SU(N|M)$ if $N \neq M$. The case $N = M$, however, requires a further study.

This fact is relevant for us because $PSU(2, 2|4)$, which is a real form of $PSU(4|4)$, is the symmetry of $\mathcal{N} = 4$ super Yang-Mills theory, which thus involves the exceptional case $N = M = 4$. (For our purposes, the signature is not important, as we will be carrying out purely algebraic manipulations; we need not distinguish $PSU(4|4)$ from $PSU(2, 2|4)$.)

First we give a few facts about the Lie superalgebras $U(n|n)$, $SU(n|n)$ and $PSU(n|n)$ (see *eg.* [15]). For references to super Yangians, see *eg.* [16–20]. The Lie superalgebra $U(n|m)$ has generators that can be represented by matrices of the form $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a and d are $n \times n$ and $m \times m$ bosonic hermitian matrices, and b and c are $n \times m$ and $m \times n$ fermionic

matrices and are hermitian conjugates. The supertrace of x is $\text{Str } x = \text{Tr } a - \text{Tr } d$. If we divide by multiples of the identity, we get a superalgebra $PU(n|m)$. If we restrict to x such that $\text{Str } x = 0$, we get a superalgebra of one less dimension that is called $SU(n|n)$. For $n \neq m$, at the Lie algebra level, requiring the trace of x to be zero removes the identity matrix and hence $PU(n|m)$ and $SU(n|m)$ are the same at the Lie algebra level (the global structure of the groups is different). For $n = m$, the identity matrix has zero supertrace, so requiring the trace to be zero does not remove the identity matrix. If we require x to be traceless and further identify any two x 's that differ by an additive scalar, we get a superalgebra that is called $PSU(n|n)$ or $A(n-1|n-1)$ and has two dimensions less than $U(n|n)$.

The generators of $PSU(n|n)$ can be represented by matrices $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $\text{Tr } a = \text{Tr } d = 0$. The bosonic part of $PSU(n|n)$ is $SU(n) \times SU(n)$, generated by a and d . We are here writing x in what we may call the $n|n$ representation. In any representation, the fermionic generators of $PSU(n|n)$, here represented by the matrices b and c , transform as $n \otimes \bar{n} \oplus \bar{n} \otimes n$ under $SU(n) \times SU(n)$.

The superconformal symmetry group of $\mathcal{N} = 4$ super Yang-Mills theory is a real form of $PSU(4|4)$, whose bosonic part is $SU(4) \times SU(4)$. It will be helpful to compare $PSU(4|4)$ to its cousins $SU(4|4)$ and $U(4|4)$. We have for the Lie algebras

$$\begin{aligned} SU(4|4) &= PSU(4|4) \oplus R \\ U(4|4) &= PSU(4|4) \oplus K \oplus R. \end{aligned} \quad (35)$$

(This is an additive decomposition of the Lie algebras; the commutation relations do not preserve this decomposition.) K is the Lie algebra generated by the identity matrix (which we also write as K). R is the Lie algebra of a $U(1)$ R -symmetry group that is not contained in $PSU(4|4)$ and is not a symmetry of $\mathcal{N} = 4$ super Yang-Mills theory; we also call its generator R . In the $4|4$ representation, we take $Q_R = R$ where

$$R = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (36)$$

So commutation with R multiplies the blocks b and c of a generator x by 1 or -1 and annihilates a and d . The supertraces are

$$\text{Str } K^2 = \text{Str } R^2 = 0, \quad \text{Str } RK = 4. \quad (37)$$

We define the $U(4|4)$ structure constants by

$$[J_A, J_B] = f_{AB}^C J_C, \quad (38)$$

where the brackets denote either commutators or anticommutators. Then $f_{KB}^A = 0$ for all A, B since K is central and commutes with everything, and $f_{AB}^R = 0$ for all B, C , since the $U(1)$ R -symmetry generator R never appears on the right hand side of the commutation relations. (It is precisely because R never appears on the right hand side of the commutation relations that there can exist a theory, such as $\mathcal{N} = 4$ super Yang-Mills theory, that has $PSU(4|4)$ symmetry but not the additional $U(1)$ R -symmetry.)

The formula (38) is the first formula in this paper in which it is important to carefully distinguish whether the “ A ” index of a Lie algebra generator such as J_A is “down” or “up.” At the outset of these notes, we merely asserted that there is an invariant, nondegenerate metric g that is used to raise and lower indices, and in many formulas we have done so without comment. In the present example, we can take the metric for $U(4|4)$ to be $g_{AB} = \frac{1}{2}\text{Str } J_A J_B$. So $g_{KK} = g_{RR} = 0$, $g_{KR} \neq 0$; and when $A \neq K, R$ then $g_{AB} = 2\delta_{AB}$ and $g_{KA} = g_{RA} = 0$. It follows that when we raise and lower indices, K and R are exchanged, so the assertions in the last paragraph become

$$f_A{}^{RB} = 0 = f_K{}^{AB}. \quad (39)$$

For $U(4|4)$, the analysis in the last section applies and shows that the simple bilinear formula (9) gives a representation of the Yangian algebra as long as the single-spin representation \mathcal{R} has the property that the adjoint representation only appears once in the decomposition of $\mathcal{R} \otimes \overline{\mathcal{R}}$. For $\mathcal{N} = 4$ super Yang-Mills theory, we take \mathcal{R} to be the representation consisting of the one-particle states of the free vector multiplet. The $U(1)$ R -symmetry generator R does act on this representation (though it is not a symmetry of the gauge theory), and we take K to act on the representation by $K = 0$. In this way we interpret the representation \mathcal{R} as a representation of the extended group $U(4|4)$.

This representation does have the property that the adjoint representation only appears once in the decomposition of $\mathcal{R} \otimes \overline{\mathcal{R}}$. So we can use the familiar bilinear formula to get a multi-spin representation of the Yangian of $U(4|4)$:

$$\begin{aligned} Q_C &= g_{CC'} Q^{C'} = g_{CC'} f_{AB}^{C'} \sum_{i < j} J_i^A J_j^B \\ &= g_{CC'} f_{AB}^{C'} g^{AA'} g^{BB'} \sum_{i < j} (J_i)_{A'} (J_j)_{B'}. \end{aligned} \quad (40)$$

Here we have been careful in raising and lowering of indices to ensure that

the second generator Q_C of the Yangian transforms like the $PSU(4|4)$ generators J_C .

Since we actually want to represent the Yangian of $PSU(4|4)$, not that of $U(4|4)$, we need a few more observations. From (40), we see that in the representation of the $PSU(4|4)$ Yangian, $Q_K = 0$ and $Q_R = 2f_{AB}^K g^{AA'} g^{BB'} \sum_{i < j} (J_i)_{A'} (J_j)_{B'}$. What about the generators Q_C where C corresponds to a generator of $PSU(4|4)$? They do not depend on K , since $K = 0$ in our chosen representation, and they do not depend on R , since $f_{KB}^{C'} = 0$. It follows that the Q_C 's are given by the same formula as if we had evaluated the bilinear formula (9) directly for $PSU(4|4)$. We have thus established that this bilinear formula does give a representation of the Yangian algebra for $PSU(4|4)$.

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